Lecture 6

Monotonicity

Maximum Principles and Comparison

(a) Protter and Weinberger (1967) (classical solutions)
(b) Gilbarg and Trudinger (1977) (weak solutions)
(c) Leung 1989
Pao 1992 (monotone iteration approach to reaction-diffusion models)
(d) Hirsch 1988 b
Hess 1991
Smith 1995 (monotone dynamical system approach to reaction-diffusion models)

Theorem 6.1 (Protter and Weinberger 1967) (CC Thm 1.16)

Suppose that L has form (1.3), is strongly uniformly elliptic, with $c(x) \leq 0$. Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and that the coefficients of L are uniformly bounded on Ω

(i) Suppose that $u \in C^2(\Omega)$ and $Lu \geq 0$ in Ω. If u attains a maximum $M \geq 0$ at any point in the interior of Ω then $u(x) \equiv M$ in Ω.

(ii) Suppose further that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and that each point on $\partial\Omega$ lies on the boundary of some ball contained in Ω. If

If $u(x) = M$ at some point $x_0 \in \partial\Omega$ for which $\nabla u \cdot \vec{\eta}$ exists, then $\nabla u \cdot \vec{\eta} > 0$ at x_0 or $u(x) \equiv M$ in Ω.

Notes: (i) The geometric condition on $\partial\Omega$ in (ii) holds if $\partial\Omega$ is of class $C^{2+\alpha}$.

(ii) Corresponding results hold in the case of a minimum ≤ 0 when $Lu \leq 0$ where in (ii) $\nabla u \cdot \vec{\eta} < 0$ at x_0 or $u(x) \equiv M$ in Ω.

Consequences:

(i) Suppose $u_1, u_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ solve $Lu = f(x)$ in Ω with $u_1 \equiv u_2$ on $\partial\Omega$. If L satisfies the hypotheses of Thm 6.1, $u_1 - u_2$ cannot have a positive maximum in Ω, for then $u_1 - u_2 \equiv M$ on $\Omega \Rightarrow u_1 - u_2 \equiv M$ on $\partial\Omega$ (x) since $M > 0$. So $u_1 - u_2 \leq 0$ in Ω. Similarly, $u_1 - u_2$ cannot have a negative minimum inside Ω, so $u_1 - u_2 \geq 0$.
$\therefore u_1 \equiv u_2$.

(ii) Suppose $u_1, u_2 \in C^{2+\alpha}(\bar{\Omega})$ with $Lu_1 = f = Lu_2$, in Ω with

$$\gamma(x) u_1 + \beta \nabla u_1 \cdot \eta = \gamma(x) u_2 + \beta \nabla u_2 \cdot \eta \quad \text{on } \partial\Omega.$$

Then $L(u_1 - u_2) = 0$ in Ω

$$\gamma(x)(u_1 - u_2) + \beta \nabla(u_1 - u_2) \cdot \eta = 0 \quad \text{on } \partial\Omega$$

Suppose $u_1 - u_2$ has a positive maximum M in the interior of Ω. Then $u_1 - u_2 \equiv M$ in Ω and hence on $\bar{\Omega} \Rightarrow \nabla(u_1 - u_2) \equiv 0$ on $\bar{\Omega}$

$$\Rightarrow \gamma(x)(u_1 - u_2) \equiv 0 \quad \text{on } \partial\Omega \quad (x)$$

since $\gamma(x) \not\equiv 0$ on $\partial\Omega$.

If $u_1 - u_2$ has a positive maximum on $\partial\Omega$ at say x_0,

since $\beta(x_0)\nabla(u_1 - u_2)(x_0)\cdot\eta = -\gamma(x_0)(u_1 - u_2)(x_0) \leq 0$,

$u_1 - u_2 \equiv M$ on Ω, and we argue as before.

So $u_1 - u_2 \leq 0$ on $\bar{\Omega}$.

We may now argue in an analogous fashion

that $u_1 - u_2$ cannot obtain

a negative minimum in $\bar{\Omega}$. $\therefore u_1 - u_2 \geq 0$. $\therefore u_1 - u_2 \equiv 0$.

(iii) Suppose $-Lu \geq 0$ in Ω. Then $Lu \leq 0$. So u can not have a nonpositive minimum M inside Ω unless $u \equiv M$. If $u \geq 0$ on $\partial\Omega$, then we must have $u \geq 0$ on Ω, with $u > 0$ in Ω unless $u \equiv 0$.

If now $Lu_1 \leq Lu_2$ in Ω and $u_1 \geq u_2$ on $\partial\Omega$,

then $-L(u_1 - u_2) \geq 0$ in Ω and $u_1 - u_2 \geq 0$ on $\partial\Omega$.

So $u_1 \geq u_2$ on Ω.

(iv) Suppose u is an equilibrium of a diffusive logistic equation

$$0 = \nabla \cdot d(x) \nabla u + r[1 - (u/K)]u \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega.$$

If u is continuous on $\bar{\Omega}$, u attains a maximum somewhere in $\bar{\Omega}$. If u has a maximum $M > K$ at some point $x^* \in \Omega$ then for $x = x^*$ we must have $\frac{\partial u}{\partial x_i} = 0$ and $\frac{\partial^2 u}{\partial x_i^2} \leq 0$ for $i = 1, \dots, n$.

Now $$\nabla \cdot d(x) \nabla u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(d(x) \frac{\partial u}{\partial x_i}\right)$$

$$= \sum_{i=1}^{n} \left[d(x) \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial d}{\partial x_i} \cdot \frac{\partial u}{\partial x_i}\right]$$

$$= d(x) \Delta u + \nabla d \cdot \nabla u$$

At x^*, $\nabla \cdot d(x) \nabla u \leq 0$.

So $0 \leq -\nabla \cdot d(x) \nabla u$

$$= r\left[1 - \frac{M}{K}\right] M < 0 \qquad (\times)$$

So $u \leq K$ for $x \in \Omega$.

The maximum principle extends to parabolic equations. Here some results no longer require the hypothesis $c \leq 0$.

Theorem 6.2 Suppose L has the form (1.3) (allowing coefficients to depend upon t.), L is strongly uniformly elliptic (for $(x,t) \in \bar{\Omega} \times [0,T]$), $c(x,t) \leq 0$, and the coefficients of L are uniformly bounded on $\Omega \times [0,T]$

(i) Suppose that $u \in C^{2,1}(\Omega \times (0,T])$ and that $\frac{\partial u}{\partial t} - Lu \leq 0$ in $\Omega \times [0,T]$. If u attains a maximum $M \geq 0$ at a point $(x_0,t_0) \in \Omega \times (0,T]$, then $u(x,t) \equiv M$ on $\Omega \times (0,t_0]$

(ii) Suppose further that $u \in C^{2,1}(\Omega \times [0,T]) \cap C(\bar{\Omega} \times (0,T])$ and that each point of $\partial\Omega$ lies on the boundary of some ball inside Ω. If $u(x_0,t_0) = M$ at some point of $\partial\Omega \times (0,T]$ for which $\frac{\partial u}{\partial \eta}$ exists, then either $\frac{\partial u}{\partial \eta} > 0$ at (x_0,t_0) or $u \equiv M$ on $\bar{\Omega} \times (0,t_0]$.

Note: If $u_t - Lu \geq 0$, u cannot attain a minimum $M \leq 0$

at $(x_0, t_0) \in \Omega \times (0, T]$ unless $u \equiv M$ on $\Omega \times [0, t_0]$, and similarly, if $u(x_0, t_0) = M$ at a point $(x_0, t_0) \in \partial\Omega \times (0, T]$, either $\partial u / \partial \eta < 0$ at (x_0, t_0) or $u \equiv M$ in $\Omega \times (0, t_0]$.

Corollary 6.3 (Cor 1.18) Suppose L and $\Omega \times (0, T]$ satisfy the hypotheses of Theorem 6.2, save for the requirement $c(x, t) \leq 0$.

Suppose $\gamma(x)$ and $\beta(x)$ are bounded functions on $\partial\Omega$ with $\gamma(x) \geq 0$ and $\beta(x) > 0$.

If $u(x, t) \in C^{2,1}(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T])$ with

$$u_t - Lu \geq 0 \text{ on } \Omega \times (0, T), \quad u(x, 0) \geq 0 \text{ on } \Omega$$

$\boxed{\text{and}}$

$$\gamma(x) u(x, t) + \beta(x) \frac{\partial u}{\partial \eta}(x, t) \geq 0 \quad \text{on } \partial\Omega \times (0, T]$$

$\boxed{\text{or}}$

$$u \geq 0 \quad \text{on } \partial\Omega \times (0, T],$$

then either $u(x, t) > 0$ on $\Omega \times (0, T]$

or $u(x, t) \equiv 0$ on $\bar{\Omega} \times [0, t_0]$ for some $t_0 > 0$.

If $u(x, 0) > 0$ for some $x \in \Omega$, or if there $\exists$ a $t_1 > 0$ such that for each $t \in (0, t_1)$,

either $u(x, t) > 0$ or $\gamma u(x, t) + \beta \frac{\partial u}{\partial \eta}(x, t) > 0$

for some $x \in \partial\Omega$, then $u(x,t) > 0$ in $\Omega \times (0,T]$.

Pf. Suppose k is large enough so that $c - k \leq 0$.

Set $w = e^{-kt} u$.

Then $w_t - (Lw - kw) = -kw + e^{-kt} u_t - e^{-kt} Lu + kw$

$$= e^{-kt}(u_t - Lu) \geq 0$$

So Theorem 6.2 applies to w.

If $w < 0$, it attains a negative minimum on $\bar{\Omega} \times (0,T]$.

If this minimum is attained in $\Omega \times (0,T]$, Thm 6.2 (i)

$\Rightarrow$ $w \equiv M$ on $\Omega \times (0, t_0]$ for some $t_0 > 0$

$\Rightarrow$ $w \equiv M$ on $\bar{\Omega} \times [0, t_0]$ $\Rightarrow$ $u(x,0) < 0$ ⊗

If the minimum is attained on $\partial\Omega \times (0,T]$, Thm 6.2 (ii)

$\Rightarrow$ $\frac{\partial w}{\partial n} < 0$ at the point $\Rightarrow$ $\gamma(x) w(x,t) + \beta(x) \frac{\partial w}{\partial n}(x,t) < 0$

at the point ⊗

$\boxed{\text{or}}$ that $w(x,t) \equiv M$ on $\Omega \times (0, t_0]$, which

leads to the same contradiction as before.

So $w \geq 0$ on $\bar{\Omega} \times [0,T]$ $\Rightarrow$ $u(x,t) \geq 0$ on $\bar{\Omega} \times [0,T]$

If $w = 0$ at any point $(x,t) \in \Omega \times (0,T]$, Thm 6.2 $\Rightarrow$ $w \equiv 0$ on

$\bar{\Omega} \times [0,t_0] \Rightarrow u \equiv 0$ on $\bar{\Omega} \times [0,t_0]$.

If $u(x,0) > 0$ for some $x \in \Omega$ or

either $u(x,t) > 0$ or $\gamma u + \beta \frac{\partial u}{\partial \eta} > 0$ for some

$x \in \partial\Omega$ for $t \in [0,t_1]$, then u cannot be

identically zero on $\bar{\Omega} \times [0,t_0]$, so we must

have $w > 0$ and hence $u > 0$ on $\Omega \times (0,T]$.

Theorem 6.4 (CC Thm 11.19) Suppose L and Ω satisfy the hypotheses of Theorem 6.2 with $c(x,t) \equiv 0$.

Suppose that $f(x,t,u)$ and $\frac{\partial f}{\partial u}(x,t,u) \in C(\bar{\Omega} \times [0,T] \times \mathbb{R})$.

Suppose $\bar{u}, \underline{u} \in C^{2,1}(\Omega \times (0,T]) \cap C(\bar{\Omega} \times [0,T])$

with

(6.1) $$\frac{\partial \bar{u}}{\partial t} - L\bar{u} \geq f(x,t,\bar{u}) \quad \text{in } \Omega \times (0,T]$$

(6.2) $$\frac{\partial \underline{u}}{\partial t} - L\underline{u} \leq f(x,t,\underline{u}) \quad \text{in } \Omega \times (0,T]$$

$\bar{u}(x,0) \geq \underline{u}(x,0)$ on Ω, and either

$$\bar{u}(x,t) \geq \underline{u}(x,t)$$

or

$$\gamma(x)\bar{u} + \beta(x)\frac{\partial \bar{u}}{\partial \eta} \geq \gamma \underline{u} + \beta(x)\frac{\partial \underline{u}}{\partial \eta} \qquad (\gamma \geq 0,\ \beta > 0)$$

on $\partial\Omega \times (0,T]$,

then either $\bar{u} \equiv \underline{u}$ or $\bar{u} > \underline{u}$ on $\Omega \times (0,T]$

Pf: Set $u = \bar{u} - \underline{u}$

$$\frac{\partial u}{\partial t} - Lu - c(x,t)u \geq 0$$

where $c(x,t) = \dfrac{[f(x,t,\bar{u}) - f(x,t,\underline{u})]}{\bar{u} - \underline{u}}$

is bounded since $\frac{\partial f}{\partial u}$ is continuous.

Protter and Weinberger (1967)
Walter (1970)
Fife (1979)
Smoller (1982)
Leung (1989)
Pao (1992)

Such comparison theorems are the basis for applying monotone dynamical systems to reaction-diffusion models.

Hirsch (1988)
Smith (1995)

For systems we have,

(CC Thm 1.20)

Theorem 6.5 Suppose that Ω and the operators L_i, $i=1,\dots,m$ satisfy the conditions of Theorem 6.2 with $c_i \equiv 0$ for each i.

Suppose that for each i the functions $f_i(x,t,\vec{u})$ and $\frac{\partial f_i}{\partial u_j}(x,t,\vec{u})$, $i,j=1,\dots,m$, belong to $C(\bar{\Omega}\times[0,T]\times\mathbb{R}^m)$ and that

$$\frac{\partial f_i}{\partial u_j} \geq 0 \quad \text{if } i \neq j$$

Then if $\vec{w}=(w_1,\dots,w_m)$ and $\vec{v}=(v_1,\dots,v_m)$ satisfy

$$\frac{\partial w_i}{\partial t} - L_i w_i \geq f_i(x,t,\vec{w})$$
$$\frac{\partial v_i}{\partial t} - L_i v_i \leq f_i(x,t,\vec{v}) \qquad \text{in } \Omega \times (0,T]$$

with $w_i(x,0) \geq v_i(x,0)$

and <u>either</u>

$$w_i \geq v_i \quad \text{on } \partial\Omega \times (0,T]$$

<u>or</u>

$$\gamma_i(x) w_i + \beta_i(x) \frac{\partial w_i}{\partial \eta} \geq \gamma_i(x) v_i + \beta_i(x) \frac{\partial v_i}{\partial \eta} \quad \text{on } \partial\Omega \times (0,T],$$

$i = 1, \dots, m \quad (\gamma_i \geq 0,\ \beta_i > 0),$

$w_i \geq v_i$ in $\Omega \times (0,T]$ for $i = 1, \dots, m$.

Notes: (i) $\dfrac{\partial f_i}{\partial u_j} \geq 0$ is called a quasimonotone condition. Such systems are called <u>cooperative</u>.

(ii) It is possible to have $w_i > v_i$ in some components but $w_j \equiv v_j$ in others.

(iii) Many models for two competing species still admit comparison principles. Suppose that u_1 and u_2 satisfy

$$\frac{\partial u_i}{\partial t} - L_i u_i = f_i(x, t, u_1, u_2), \quad i = 1, 2$$

with $\frac{\partial f_1}{\partial u_2} \le 0$ and $\frac{\partial f_2}{\partial u_1} \le 0$, where L_i is as

in Theorem 6.5. Now let $\hat{u}_2 = k - u_2$ for some constant k.

$$\frac{\partial u_1}{\partial t} - L_1 u_1 = \tilde{f}_1(x, t, u_1, \tilde{u}_2) = f_1(x, t, u_1, k - \tilde{u}_2)$$

$$\frac{\partial \hat{u}_2}{\partial t} - L_2 \hat{u}_2 = \hat{f}_2(x, t, u_1, \hat{u}_2) = -f_2(x, t, u_1, k - \hat{u}_2)$$

Hence $\frac{\partial \hat{f}_1}{\partial \tilde{u}_2} = -\frac{\partial f_1}{\partial u_2} \ge 0$ and $\frac{\partial \tilde{f}_2}{\partial u_1} = -\frac{\partial f_2}{\partial u_1} \ge 0$

So we can apply Theorem 6.5 to the converted system

and conclude that if (w_1, w_2) and (v_1, v_2) are

solutions with $w_1 \ge v_1$, $w_2 \le v_2$ for $t = 0$,

then $w_1 \ge v_1$ and $w_2 \le v_2$ for all $t > 0$.

We refer to $\underline{u}$, $\vec{v}$ as sub (or lower) solutions and

$\bar{u}$, $\vec{w}$ as super (or upper) solutions.

(CC Prop. 1,21)

Proposition 6.6 Suppose that L, Ω and f satisfy the assumptions of Theorem 6.5. Suppose that L and Ω also satisfy the hypotheses of Theorem 5.3 (in the notes, see Prof. Zhao's webpage) and that $f(x,t,u)$ is Hölder continuous with exponent α wrt x and exponent $\alpha/2$ wrt t. Suppose that $h(x) \in C_0^{2+\alpha}(\bar{\Omega})$ with

$$Lh = f(x,0,0) \quad \text{on } \partial\Omega,$$

and that $\bar{u}$, $\underline{u}$ satisfy (6.1), (6.2), respectively, with

$$\underline{u}(x,0) \le h(x) \le \bar{u}(x,0) \quad \text{on } \bar{\Omega}$$

and

$$\underline{u}(x,t) \le 0 \le \bar{u}(x,t) \quad \text{on } \partial\Omega \times [0,T].$$

Then the problem

$$\frac{\partial u}{\partial t} - Lu = f(x,t,u) \quad \text{in } \Omega \times (0,T]$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T]$$

$$u(x,0) = h(x) \quad \text{on } \bar{\Omega}$$

has a solution $u^* \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [0,T])$

with $\underline{u}(x,t) \le u^*(x,t) \le \bar{u}(x,t)$.

Note: The coefficients of L here may depend on t, which is allowed in Theorem 5.3.
(In the applications we discuss, they do not.)

Sketch of proof: Choose a constant C large enough so that

$$\frac{\partial f}{\partial u} + C \ge 0$$

on $\bar{\Omega} \times [0,T]$ for $\underline{u}(x,t) \le u \le \bar{u}(x,t)$

Let $\underline{u}^0 = \underline{u}$ and $\bar{u}^0 = \bar{u}$.

Define $\underline{u}^k$ and $\bar{u}^k$ recursively as the solutions of

$$\frac{\partial u}{\partial t} - Lu + Cu = f(x, t, \underline{u}^{k-1}) + C\underline{u}^{k-1} \quad \text{in } \Omega \times (0,T]$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T]$$

$$u(x,0) = h(x) \quad \text{in } \overline{\Omega}$$

and

$$\frac{\partial u}{\partial t} - Lu + Cu = f(x, t, \overline{u}^{k-1}) + C\overline{u}^{k-1} \quad \text{in } \Omega \times (0,T]$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T]$$

$$u(x,0) = h(x) \quad \text{on } \overline{\Omega}.$$

Set $v^k = \underline{u}^k - \underline{u}^{k-1}$

Then one may show via induction that $v^k \geq 0$ for $k \geq 1$, so that $\underline{u}^k \geq \underline{u}^{k-1}$ for $k \geq 1$.

One may also show that $\overline{u}^{k+1} \leq \overline{u}^k$, and that for each k, $\underline{u}^k \leq \overline{u}^k$.

Since $\{\underline{u}^k\}$ is increasing and bounded above by $\overline{u}^0$

$\{\underline{u}^k\}$ converges pointwise. One then uses regularity theory to argue that the limit function is a solution.

Theorem 6.7 (CC Thm 1.22) Suppose L has the form (1.3) and is strongly uniformly elliptic and that L and Ω satisfy the hypotheses of Theorem 4.1, but without the restriction $c \leq 0$. Suppose that the coefficients a_{ij} of L are uniformly Lipschitz in $\bar{\Omega}$ and that $f(x,u)$ and $\frac{\partial f}{\partial u}(x,u)$ are Hölder continuous in x and continuous in u on $\bar{\Omega} \times \mathbb{R}$. Finally, suppose that $\bar{u}$ and $\underline{u}$ are super- and sub-solutions to the problem

$$Lu + f(x,u) = 0 \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

i.e.
$$-L\bar{u} \geq f(x,\bar{u})$$

$$-L\underline{u} \leq f(x,\underline{u})$$

with $\underline{u} \le 0 \le \bar{u}$ on $\partial\Omega$

and $\underline{u} \le \bar{u}$ on Ω.

Let v be the solution to

$$(6.3) \quad \begin{cases} \dfrac{\partial v}{\partial t} = Lv + f(x,v) & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = \underline{u}(x) \end{cases}$$

Then $v(x,t)$ is monotonically increasing with respect to t, and as $t \to \infty$, $v(x,t)$ converges to an equilibrium of (6.3) which is the minimal equilibrium that satisfies $\underline{u}(x) \le u^*(x) \le \bar{u}(x)$.

If w satisfies (6.3) with $w(x,0) = \bar{u}(x)$, then $w(x,t)$ is monotonically decreasing with respect to t and converges to $u^{**}(x)$, the maximal equilibrium for (6.3) which satisfies

$$\underline{u}^*(x) \le u^{**}(x) \le \bar{u}(x).$$